The Method of Variables Separable for Finding the Exact Solutions of Pdes

Soe Soe Moe
University of Computer Studies (Loikaw)
ssmoe7@gmail.com

Abstract

In this paper, we introduce one-dimensional wave equation. We can use many methods solving partial differential equations. Among them are Euler's method, Taylor series method, Runge-Kutta method, Picard's method and so on. The method of separation of variables and Fourier series method are use. In example 2.1 and article 3, we describe the solutions of second order and fourth order partial differential equation and these exact solutions are usefully interpreted in terms of the diagram.

1. Introduction

The wave equation is untypically among partial differential equations in that it is possible to write down all the solutions. Then we will solve the wave equation by finding a particular solution and the constructing the general solution. A method of solution to the wave equation will be calculated using the separation of variables and the Fourier series expansion. Matlab software is used to develop this paper.

2. Second Order Partial Differential Equation

Consider the wave equation:

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{\pi}{\rho}$,

$u(x,t) = 0, u(L,t) = 0$, for all $t \geq 0$ and (1)

$u(0,t) = f(x),\ u_t(x,0) = g(x), 0 \leq x \leq L.$

Assume that the general solution of "(1)" is

$u(x,t) = F(x)\, G\,(t)$ (2)

where F and G are functions of x and t respectively.

Therefore we get $\frac{\partial^2 u}{\partial t^2} = FG''$ and $\frac{\partial^2 u}{\partial x^2} = F''G$.

Rewrite "(1)" in terms of F and G such that $FG'' = c^2 F''G$. Then we get $\frac{G''}{c^2 G} = \frac{F''}{F}$. Hence the left sight depending only on t and right side only on x. Hence both sides must be constants. Suppose that

$\frac{G''}{c^2 G} = \frac{F''}{F} = k.$ (3)

Therefore $G'' - c^2 kG = 0$ and (4)

$F'' - k\,F = 0$ (5)

where the separation constant k is arbitrary. The nature of the solution of "(3)" depends upon the value of k. We consider the values of k are zero, positive number and negative number. We calculate the solutions of F and G of "(4)" and "(5)" by using the given initial and boundary conditions.

Since $u(x,t) = 0, u(L,t) = 0$, we get

$F(0)G(t) = 0$ and $F(L)G(t) = 0$ for all $t \geq 0$. (6)

If $G(t) = 0$ then $u = F(x)G(t) = 0$.

Hence $G(t) \neq 0$ and then by "(6)"

$F(0) = 0$ and $F(L) = 0.$ (7)

For $k = 0$, the general solution of "(5)" is $F = ax + b$ and then from "(7)" we get $a = b = 0$ and so that $F = 0$ and it gives $u = FG = 0$

For $k = \mu^2 > 0$, the general solution of "(5)" is $F\,(x) = A\, e^{\mu x} + Be^{-\mu x}$. By using the given conditions, we get $F = 0$ and hence $u = 0$.

For $k = -\mu^2 < 0$, the general solution of "(5)" is $F\,(x) = A\cos\mu\, x + B\sin\mu x$.

By using given conditions, we get

$A = 0$ and $B\sin\mu\, L = 0.$

We must take $B \neq 0$.

Hence $\sin\mu\, L = 0$ .Thus $\beta\, L = n\,\pi$, where n is a positive integer. Choosing $B = 1$, we obtain infinitely many solutions such that

$F\,(x) = F_n(x), n = 1,2,\ldots$.

Hence $F_n(x) = \sin\frac{n\pi}{L}x, n = 1,2,\ldots$.

For the value $k = -\mu^2 = -\left(\frac{n\pi}{L}\right)^2$, "(4)" becomes

$G'' + c^2\left(\frac{n\pi}{L}\right)^2 G = 0$.

$G'' + \mu^2\, G = 0$ where $\mu_n = c\,\frac{n\pi}{L}$. (8)

Thus the general solution is

$u_n(x,t) = F_n\,(x,t)G_n(x,t)$ satisfies "(2)".

Therefore we get the general solution is

$u_n(x,t) = \sin\frac{n\pi}{L}x\,(A_n\cos\beta_n t + B_n\sin\beta_n t\,)$ (9)

where $n = 1,2,\ldots$.

To obtain a solution that also satisfies the given boundary condition, we take the solution of the form

$u\,(x,t) = \sum_{n=1}^{\infty} u_n(x,t) =$

$\sum_{n=1}^{\infty}(A_n\cos\beta_n t + B_n\sin\beta_n t\,)\sin\frac{n\pi}{L}x$ (10)

$u_n(x,t) =$

$\sum_{n=1}^{\infty}(-A_n\beta_n\cos\beta_n t + B_n\beta_n\sin\beta_n t\,)\sin\frac{n\pi}{L}x.$ (11)

Using $u(0,t) = f(x),\ u_t(x,0) = g(x), 0 \leq x \leq L$ "(10)" becomes

$u_n(x,t) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi}{L}x = f\,(x).$

Here $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, n = 1,2,\ldots$. (12)

Using $u_t(x,0) = g\,(x)$ and "(11)" we also get the general solution of "(1)" is

$u\,(x,t) = \sum_{n=1}^{\infty} A_n\cos\beta_n\, t\sin\frac{n\pi}{L}x$, where $n = 1,2,\ldots$. [2],[5]

2.1. Example

Consider the wave equation:

$\frac{\partial^2 u}{\partial t^2} = 3^2\frac{\partial^2 u}{\partial x^2}$, where $3^2 = \frac{T}{\rho}$

$u(0,t) = 0,\ u(\pi,t) = 0$, for all $t \geq 0$ and $u(x,0) = sinx,\ u_t(x,0) = 1,\ 0 \leq x \leq \pi.$

Assume a solution of the form $u(x,t) = F(x)G(t)$ where F is a function of x alone and G is a function of t alone. To get the general solution $u(x,t)$ variable separable method and the Fourier series method are used. Therefore the solution is

$u(x,t) = cos3t\ sinx + \frac{2}{3\pi}\sum_{n=1}^{\infty} \left(\frac{1-cosn\pi}{n^2}\right) sin3nt\ sinx.$

The partial sum for $n = 5$ is considered.

The values of $u(x,t)$ from $t = 0\ to\ 10$ and $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\ and\ \pi$ are computed (run by matlab).[1]

Table 1. Solutions $u(x,t)$ for the wave equation

			$u(x,t)$		
t	x=0	x=$\frac{\pi}{4}$	x=$\frac{\pi}{2}$	x=$\frac{3\pi}{4}$	x = π
0	0	0.707	1	0.707	1.23e-16
1	0	-0.07	-0.098	-0.07	-1.20e-16
2	0	-0.513	-0.725	-0.513	-8.88e-17
3	0	0.099	1.402	0.099	1.718e-16
4	0	-1.355	-1.916	-1.355	-2.34e-16
5	0	1.623	2.295	1.623	2.81e-16
6	0	-1.835	-2.596	-1.835	-3.18e-16
7	0	2.031	2.872	2.031	3.52e-16
8	0	-2.230	-3.154	-2.230	-3.86e-16
9	0	2.429	3.436	2.429	4.21e-16
10	0	-2.608	-3.688	-2.608	-4.52e-16

Now, we show by diagram with these solutions.[2], [5]

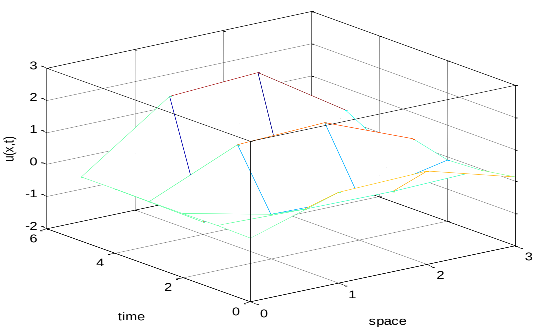

Fig1. The solutions $u\ (x,t)$ with 2D

3.Fourth Order partial differential Equation

In this section, we find the solutions of partial differential equation with their given conditions. Solutions $u\ (x\ ,t)$ run by matlab programming. Here we computed the partial sum for $n = 1\ to\ 5$ .Consider the fourth order partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = 9\ \frac{\partial^4 u}{\partial x^4} \qquad (13)$$

$u(0,t) = 0, u_{xx}(0,t) = 0\ ,$

$u(x,0) = f(x) = 1 - x\ , u_t(x\ ,0) = g(x) = \ 0,$

$for\ all\ t \geq 0$ and $0\ \leq x\ \leq 1.$

Assume the general solution of "(13)" is of the form

$u(x,t) = F(x)G(t)\ ,$ (14)

where $F\ and\ G$ are functions of x and t respectively. Rewrite the "(13)" of the form is

$F\ G'' \ = \ -9\ F''''\ G$. Then we get

$$\frac{G''}{-9\ G} = \frac{F''''}{F}\ .$$

Hence the left side depending only on t and the right side only on x. Hence both sides must be constants. Suppose that $\frac{G''}{-9\ G} = \frac{F''''}{F} = \lambda$ where $\lambda\ \geq 0$. (15)

Therefore

$G''\ +\ 9\ \lambda\ G = 0$ and (16)

$F''''\ -\ \lambda F\ = 0$ where $t \geq 0$ and $0\ \leq x\ \leq 1.$ (17)

The nature of the solution of "(13)" depends upon the value of λ.

Now, we calculate the solutions of F and G by using the given boundary and initial conditions.

Hence $\lambda = 0$, "(17)" becomes

$F\ (x) = A\ x^3 +\ B\ x^2 +\ Cx + D,$ where $A, B, C\ and\ D$ are arbitrary constants.

Therefore $F'\ = 3Ax^2 + 2\ Bx + C$ and

$F''\ = 6\ Ax + 2B$.We consider in the case $\lambda = 0$.

Hence we obtain

$F\ (0) = D\ = 0$ and $F''(0) = 2\ B = 0$

And then $F''(1) = 6\ A = 0$ and $F\ (1) = 0.$

Therefore we conclude that the solution of

$F\ = 0$.Hence $\lambda = 0$ can not be an eigenvalue.

For $\lambda > 0\ , G = a\cos\left(3\ \sqrt{\lambda}\ t\right) +\ b\sin\left(3\ \sqrt{\lambda}\ t\right).$

Let $F\ (x)\ =\ e^{\alpha x}$ and substituting this into equation "(17)",$\alpha^4 - \lambda = 0$ and hence $\alpha\ =\ \pm(\ \lambda^{1/4}\ ).$

Now, we suppose that $\mu =\ \ \lambda^{1/4}$. Then

$F\ \ = A\cos(\mu x) +\ B\ \sin(\mu x) + C\ cosh\ (\mu x) + D\ sinh\ (\mu x).$

$F'\ =\ -A\ \mu \sin(\mu x) + B\mu\ cos\ (\mu x) + C\ \mu\ sinh\ (\mu x) + D\ \mu \cosh(\mu x)$ and

$F''\ =\ -A\mu^2\cos(\mu x) -\ B\mu^2\sin(\mu x) + C\mu^2\ coshh\ (\mu x) +\ D\mu^2\sinh(\mu x)\ .$

By using given boundary condition, we obtain $F\ (0) = A + C\ = 0$ and

$F''(0) =\ -A\mu^2 +\ C\mu^2 = 0.$

From these two equations, $A\ = C\ = 0$. Also we have

$F\ (1) = B\sin\mu + D\sinh\mu = 0\ \ and$

$F''(1) =\ -B\mu^2\sin(\mu x) + D\mu^2\sinh(\mu x) = 0\ .$

From these two equations, $D\ = 0$ and $B\sin\mu\ = 0.$

Since $B\ \neq 0$ it implies that $\sin\mu = 0$ where μ is any integer. This gives $\mu = \mu_n = n\pi$, where $n = 1,2, \ldots.$

Therefore $F\ =\ F_n\ =\ \sum_{n=1}^{\infty}\sin(n\pi x).$

Now, we solve for the solution $G = a\cos\left(3\ \sqrt{\lambda}\ t\right) + b\sin\left(3\ \sqrt{\lambda}\ t\right).$

This implies that

$G_n\ =\ a_n\cos(3\ \mu^2\ t) +\ b_n\sin(3\ \mu^2\ t)\ .$

$G_n\ =\ a_n\cos(3\ n^2\pi^2\ \ t) +\ b_n\sin(3\ n^2\pi^2\ t)$, where $\mu_n = n\ \pi.$

Then the general solution of equation (13) is of the form

$u\ (x,t) =\ u_n(x,t) =\ F_n\ (x)G_n$ (t).

Therefore $u_n(x,t) =$

$\sum_{n=1}^{\infty}\{a_n\cos(3\ n^2\pi^2\ \ t) + b_n sin\ (3\ n^2\pi^2\ \ t)\}\ \sin(n\pi x).$ (18)

where $a_n\ = 2\ \int_0^1 f(x)\sin n\ \pi x\ dx$ and

$b_n = \frac{2}{3n^2\pi^2}\int_0^1 \sin(n\,\pi x)\ dx$.

Differentiating (18) with respect to t,

$u_t(x,t) = \sum_{n=1}^{\infty}\{-a_n \sin(3\,n^2\pi^2\ t)\,3n^2\,\pi^2 + b_n cos\,(3\,n^2\pi^2\ t)3\,n^2\pi^2\}\ \sin(n\pi x)$. (19)

By using boundary conditions in "(18)" it implies that

$b_n = 0$ and $a_n = 2\int_0^1 (1-x)\sin(n\pi x)\ dx.$ (20)

Now, we find the value of a_n by using the integrating by parts.

Therefore $a_n = 2\left[\frac{1}{n\pi} - \frac{1}{n^2\pi^2}\sin(n\,\pi x)\right]$. Thus the general solution of "(13)" is

$u_n(x\,,t) = \sum_{n=1}^{\infty} 2\left[\frac{1}{n\pi} - \frac{1}{n^2\pi^2}\sin(n\pi)\right]\cos(3\,n^2\pi^2 t)\,\sin(n\pi x).$

This equation is usefully interpreted in terms of the following diagram.

The solutions $u\,(x,t)$ considered the partial sum for $n = 1\ to\ 5,\ x = 1\ to\ 5\ and\ t = 1\ to\ 10$.[2], [3],[4]

Table2. Solutions $u(x,t)$ for 4th order pde

	$u\,(x,t)$				
t	x=1	x=2	x=3	x=4	x=5
1	0.044	0.000	-0.044	-0.000	0.044
2	-0.087	-0.000	0.087	0.000	-0.087
3	-0.108	-0.000	0.108	0.000	-0.108
4	0.008	0.000	-0.008	-0.000	0.008
5	0.114	0.000	-0.114	-0.000	0.114
6	0.075	0.000	-0.075	-0.000	0.075
7	-0.059	-0.000	0.059	0.000	-0.059
8	-0.118	-0.000	0.118	0.000	-0.118
9	-0.028	-0.000	0.028	0.000	-0.028
10	0.098	0.000	-0.098	-0.000	0.098

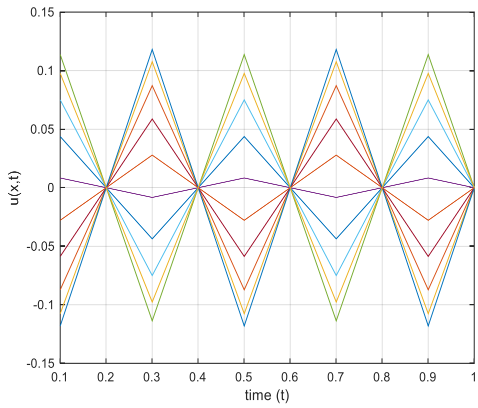

Fig 2. The solutions $u\,(x,t)$ with 2D

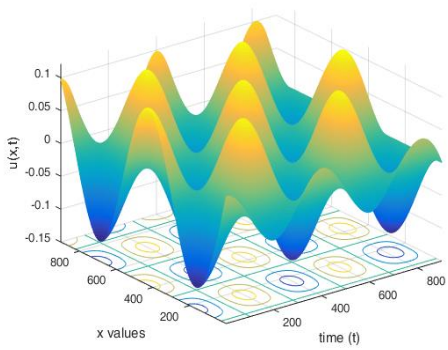

Fig 3. The solutions $u\,(x,t)$ with 3D

4. Conclusion

In this paper, we focus our attention on the variables separable method and Fourier series method to drive exact solutions of the partial differential equations. We have considered one dimensional wave equation with appropriate boundary and initial conditions. A general solution is obtained by taking an arbitrary linear combination of homogeneous equation. We applied the variable separable method to get the exact solutions of second order wave equation and fourth order equation. These solutions are usefully interpreted in terms of the diagram. Finally, this method is productive, effective and well-built mathematical tool for solving partial differential equations.

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