Some Applications of Laplace Transforms for Electric Circuits

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Abstract

Laplace transform is a technique for solving differential equations. Laplace transform is a mathematical tool which is used in solving the time domain function by converting it into frequency domain function. In this paper, applications of Laplace transform in science and engineering fields are discussed. Some basic properties of Laplace transform and inverse Laplace transform are also presented. Then, solving differential equations and the currents by using Laplace transforms are discussed.

Keywords: Laplace transform, Inverse Laplace transform, Kirchhoff’s Voltage Law, Differential equations, RC/RL circuits.

1. Introduction

The Laplace transform is applied in different areas of science, engineering and technology. Laplace transform is applicable in several fields. Laplace transforms are invaluable for any engineer’s mathematical toolbox as they make solving linear ordinary differential equations and related initial value problems. Transformation in mathematics deals with the conversion of one function to another function that may not be in the same domain. Laplace transform is named in honour of the great mathematician and renowned astronomer Pierre Simon Marquis De Laplace (1749-1827) who lived in France. The Laplace transformation is an important part of control system engineering. Laplace transform of the different functions (function of time) are carried out. Inverse Laplace is also an essential tool in finding out the function \( f(t) \) from its Laplace form. Both inverse Laplace transform and Laplace transform have certain properties in analyzing dynamic control systems. The Laplace transform provides a method of analyzing a linear system using algebraic methods. The analysis of electric circuits to find current is simplified by the use of Laplace transforms.

The remaining parts of this paper are arranged as follows. In section 2, some basic properties of Laplace transform are expressed. Laplace transform examples are described in section 3. Section 4 presents applications to the electric circuits and the conclusion of this paper is mentioned in section 5.

1.1. Definition of Laplace Transform

Let \( f(t) \) be the function of \( t \), time for all \( t \geq 0 \). The Laplace transform is an integral transformation of a function \( f(t) \) from the time domain into the complex frequency domain \( F(s) \). In mathematically, the Laplace transform \( \mathcal{L} \) of \( f(t) \) is given by

\[
\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt,
\]

where \( \mathcal{L} \) is Laplace transform operator.

1.2. Definition of Inverse Laplace Transform

The time function \( f(t) \) is obtained back from the Laplace transform by a process is called inverse Laplace transformation. In mathematically, inverse Laplace transform of \( F(s) \) is given by

\[
\mathcal{L}^{-1}[F(s)] = f(t).
\]

2. Some Basic Properties of Laplace Transform

This section describes some of the important basic properties and theorems of Laplace transform. They are used for solving problems in science and different engineering fields.

2.1. Linearity Property

The Laplace transform is a linear operation. Let \( \alpha \) and \( \beta \) be any two constants. If \( f(t) \) and \( g(t) \) are two functions of time \( t \), then

\[
\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)].
\]

2.2. First Shifting Theorem (s-Shifting)

If \( \mathcal{L}[f(t)] = F(s) \) (where \( s > k \) for some \( k \)), then \( e^{\alpha t}f(t) \) has the transform \( F(s - \alpha) \) (where \( s - \alpha > k \)). In formula,

\[
\mathcal{L}[e^{\alpha t}f(t)] = F(s - \alpha).
\]

2.3. Laplace Transform of Derivatives

The transforms of \( f^1(t), f^2(t), \ldots, f^n(t) \) can be described, if \( \mathcal{L}[f(t)] = F(s) \), then

\[
\mathcal{L}[f'(t)] = sF(s) - f(0),
\]

\[
\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0),
\]

and so on

\[
\mathcal{L}[f^(n)(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).
\]
2.4. Laplace Transform of Integral

Laplace transform of integral states that if \( L[f(t)] = F(s) \), then \( L[\int_0^t f(\tau) d\tau] = \frac{1}{s} F(s) \),
thus \( \int_0^t f(\tau) d\tau = L^{-1} \left( \frac{1}{s} F(s) \right) \). (6)

2.5. Unit Step Function (Heaviside Function)

The unit step function or Heaviside function \( u(t - a) \) is 0 for \( t < a \), has a jump of size 1 at \( t = a \) and is 1 for \( t > a \), in a formula:
\[
u(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases} \quad (a \geq 0). \]

2.6. Second Shifting Theorem (Time Shifting)

The second shifting theorem of Laplace transform states that if \( L[f(t)] = F(s) \), then
\[
L[f(t - a)u(t - a)] = e^{-as} F(s),
\] where \( u(t - a) \) denotes unit step function.

2.7. Initial Value Theorem

The initial value theorem of Laplace transform states that if \( L[f(t)] = F(s) \), then
\[
f(0^+) = \lim_{s \to \infty} s F(s).
\]

2.8. Final Value Theorem

The final value theorem of Laplace transform states that if \( L[f(t)] = F(s) \), then
\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s).
\]

In the following table, a few results are listed. These results can be used to find the Laplace transform.

Table 1. Table of some function \( f(t) \) and their Laplace transforms \( L[f(t)] \)

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( L[f(t)] = F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1/s )</td>
</tr>
<tr>
<td>( t )</td>
<td>( 1/s^2 )</td>
</tr>
<tr>
<td>( t^2 )</td>
<td>( 2!/s^3 )</td>
</tr>
<tr>
<td>( t^n ) (( n = 0, 1, ... ))</td>
<td>( n!/s^{n+1} )</td>
</tr>
<tr>
<td>( e^{at} )</td>
<td>( 1/(s - a) )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>( 1/(s + a) )</td>
</tr>
<tr>
<td>( \cos \omega t )</td>
<td>( s/(s^2 + \omega^2) )</td>
</tr>
<tr>
<td>( \sin \omega t )</td>
<td>( \omega/(s^2 + \omega^2) )</td>
</tr>
</tbody>
</table>

3. Laplace Transform Examples

In this section, the problem solving by using Laplace transform and inverse Laplace transform are described with some examples. The required solutions can be calculated by applying definitions and properties of Laplace transform as follows:

3.1. Example: Find the inverse Laplace transformation function of \( F(s) = \frac{4s^2 + 5s + 3}{s^3 + 4s^2 + 3s} \)\( \times \frac{1}{s+1} + \frac{3}{s+3} \)

Solution: \( F(s) = \frac{4s^2 + 5s + 3}{s^3 + 4s^2 + 3s} = \frac{4s^2 + 5s + 3}{s(s+1)(s+3)} \)

Let \( \frac{4s^2 + 5s + 3}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} \)

Thus, \( A + B + C = 4, 4A + 3B + C = 5, 3A = 3 \)

So, \( A = 1, B = -1 \) and \( C = 4 \).

Hence, \( F(s) = \frac{1}{s} + \frac{1}{(s+1)} + \frac{4}{(s+3)} \)

\( L^{-1}[F(s)] = L^{-1}\left[ \frac{1}{s} \right] + L^{-1}\left[ \frac{1}{(s+1)} \right] + L^{-1}\left[ \frac{4}{(s+3)} \right] \)

Therefore, \( f(t) = 1 - e^{-t} + 4e^{-3t} \).

3.2. Example: Find \( f(t), f'(t) \) and \( f''(t) \) for a time domain \( f(t) \). The Laplace transform form of the function is given as \( L[f(t)] = F(s) = \frac{4s+1}{2(s^2+2)} \)

Solution: Let \( L[f(t)] = F(s) = \frac{4s+1}{2(s^2+2)} \)

By applying initial value theorem, we get
\[
\begin{align*}
\lim_{t \to 0} f(t) &= \lim_{s \to \infty} s F(s) \\
&= \frac{4s+1}{2s(s^2+2)} \\
&= \frac{s}{s^2+2} = 0.
\end{align*}
\]

Now, \( L[f'(t)] = s F(s) - f(0^+) = s F(s) - 0 = s F(s) \).

Hence, \( f'(0^+) = \lim_{s \to \infty} s F(s) = \lim_{s \to \infty} s^2 F(s) = \frac{4s^2+1}{2(s^2+2)} = \frac{s^2}{s^2+2} = 2 \).

Now, \( L[f''(t)] = s^2 F(s) - s f(0^+) - f'(0^+) = s^2 F(s) - (s \times 0) - 2 = s^2 F(s) - 2 \)

Thus, \( f''(0^+) = \lim_{s \to \infty} s L[f''(t)] \)

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\[ f''(0^+)= \lim_{s \to \infty} (s^3F(s) - 2s) \]
\[ = \lim_{s \to \infty} \left( \frac{s^3}{2s^2+1} - 2s \right) \]
\[ = \lim_{s \to \infty} \frac{s^2-12s}{2s^2+6} \]
\[ = \lim_{s \to \infty} \frac{12}{2} = 0.5. \]

4. Applications to the Electric Circuits

In this section, the definitions of electric circuits such as RC-circuit and RL-circuit, Kirchhoff’s Current Law (KCL) and Kirchhoff’s Voltage Law (KVL) are presented. From these circuits, solving initial charging current and final value of steady-state current by using Laplace transform are expressed as examples.

In the theory of electrical circuits, the current flow in a capacitor is proportional to the capacitance and the rate of change in the electrical potential in SI units.

Symbolically, this is expressed by the differential equation
\[ i = C \frac{dv}{dt}, \]
where, \( C \) is the capacitance (in F(farads)) of the capacitor, \( i = i(t) \) is the electric current (in Amperes) through the capacitor as a function of time and \( v = v(t) \) is the voltage (in V(volts)) across the terminals of the capacitor, also as a function of time.

Taking Laplace transform,
\[ I(s) = C[sV(s) - V_0], \quad \text{where} \quad I(s) = L[i(t)], \]
\[ V(s) = L[v(t)], \quad V_0 = v(t) \quad \text{as} \quad t = 0. \]

Figure 1. RC-circuit

A resistor-capacitor circuit (RC-circuit) or RC-filter or RC-network (as shown in figure 1) is an electric circuit composed of resistors \( R \) (ohms) and capacitors \( C \) F(farads) driven by a voltage \( v = v(t) \) (volts) or current source. A first-order RL-circuit is composed of one resistor and one inductor and is the simplest type of RL-circuit.

The mathematical model of RC-circuit is
\[ v(t) = R \frac{di(t)}{dt} + \frac{1}{C} i(t), \]
where \( i = i(t) \) is current flowing through the RC-circuit at time \( t \).

Figure 2. RL-circuit

A resistor-inductor circuit (RL-circuit) or RL-

filter or RL-network (as shown in figure 2) is an electric circuit composed of resistors \( R \) (ohms) and inductors \( L \) (henrys) driven by a voltage \( v = v(t) \) (volts) or current source. A first-order RL-circuit is composed of one resistor and one inductor and is the simplest type of RL-circuit.

Kirchhoff’s Voltage Law (KVL) states that sum of voltages around a closed path is equal to zero.

In general, Kirchhoff’s Voltage Law (KVL) states that the electromotive force impressed on a closed loop is equal to the sum of the voltage drops across the other elements of the loop.

In mathematically,
\[ \sum_{n=1}^{N} v_n(t) = 0, \quad \text{for any loop}. \]

Take Laplace transform,
\[ \sum_{n=1}^{N} V_n(s) = 0, \quad \text{for any loop}. \]

By Kirchhoff’s Current Law (KCL) at any point of a circuit, the sum of the inflowing currents is equal to the sum of the outflowing currents.

In mathematically,
\[ \sum_{k=1}^{M} i_k(t) = 0, \quad \text{for any node}. \]

Take Laplace transform,
\[ \sum_{k=1}^{M} I_k(s) = 0, \quad \text{for any node}. \]

According to these Laws, the mathematical model of the RL-circuit is
\[ v(t) = Ri(t) + L \frac{di(t)}{dt}. \]

4.1. Example: According to figure 3, \( R = 2K \Omega \), \( C = 100 \mu F \) and \( v = 100V \), calculate the initial charging current of capacitor using Laplace transform technique.

Figure 3. RC-circuit in Example 4.1

Solution: The above figure can be redrawn in Laplace form as the following:

Figure 4. RC-circuit in example 4.1

Therefore,
\[ I(s) = \frac{V}{R+Cs} = \frac{V_{Cs}}{RCs+1} = \frac{VC}{RCs+1} \]
\[ = \frac{100 \times 100 \times 10^{-6}}{2000 \times 100 \times 10^{-6}s+1} = \frac{10^{-2}}{0.2s+1} \]
Now, initial charging current is
\[ i(0^+) = \lim_{t \to 0} i(t) = \lim_{s \to \infty} sI(s) \]
\[ = \lim_{s \to \infty} \frac{10^{-2}}{0.2s + 1} \]
\[ = \lim_{s \to \infty} \frac{10^{-2}}{0.2} = 0.05A = 50mA. \]

4.2. Example: Solve the following electric circuit, \( R = 100Ω, L = 20H \) and \( v = 100V \), by using Laplace transformation for final steady-state current.

\[ \text{Solution: By using Kirchhoff’s Voltage Law (KVL) to the circuit,} \]
\[ v(t) = Ri(t) + L \frac{di(t)}{dt} \]
\[ 100 = (100 \times i(t)) + \left( 20 \times \frac{di(t)}{dt} \right) \]
Taking Laplace both sides,
\[ \mathcal{L}[100] = 100\mathcal{L}[i(t)] + 20\mathcal{L}\left[ \frac{di(t)}{dt} \right] \]
\[ \frac{100}{s} = 100/I(s) + 20[sI(s)]. \]
As \( i(0^+) = 0, I(s) = \frac{100}{s+200}. \)
The final value of steady-state current is
\[ i(\infty) = \lim_{t \to \infty} i(t) = \lim_{s \to 0} sI(s) \]
\[ = \lim_{s \to 0} \frac{100/s}{100+20s} \]
\[ = \lim_{s \to 0} \frac{100}{100+20s} = 1A. \]

4.3. Example: Given the first order RC-circuit has equation \( 0.5 \frac{dv(t)}{dt} + v(t) = 10 \) and the initial condition of \( V(0) = 4V \), find the Laplace transform of the equation.

Solution: Let \( 0.5 \frac{dv(t)}{dt} + v(t) = 10 \).

Taking Laplace both sides,
\[ 0.5\mathcal{L}\left[ \frac{dv(t)}{dt} \right] + \mathcal{L}[v(t)] = 10\mathcal{L}[1] \]
\[ 0.5(sV(s) - V(0)) + V(s) = \frac{10}{s} \]
\[ 0.5sV(s) - 0.5(4) + V(s) = \frac{10}{s} \]
\[ (0.5s + 1)V(s) - 2 = \frac{10}{s} \]
\[ V(s) = \frac{10+2s}{s(0.5s+1)} = \frac{20+4s}{s^2+2s} \]

5. Conclusion

The paper presents the theories of Laplace transforms and examples for electric circuits using Laplace transforms are described in detail. It assists students understand the basic concepts of electric circuits. It is known that electric currents can be easily solved by means of Laplace transform method. Therefore, Laplace transform is a very effective mathematical tool to solve the complex problems in various science and engineering fields. As Laplace transforms application has no difficulty in myriad of scientific applications, many research software have made it possible to simulate the Laplace transformable equations directly which has made a good advancement in the research field.

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